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Maps, Chaos, and Fractals

Phillips Williams

Department of Mathematics and Statistics
University of Canterbury

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Phillipa Williams*
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University of Canterbury

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Abstract

The behaviour and properties of one-dimensional discrete mappings are explored by writing Matlab code to iterate mappings and draw graphs. Fixed points, periodic orbits, and bifurcations are described and chaos is introduced using the logistic map. Symbolic dynamics are used to show that the doubling map and the logistic map have the properties of chaos. The significance of a period-3 orbit is examined and the concept of universality is introduced. Finally the Cantor Set provides a brief example of the use of iterative processes to generate fractals.



*supervised by Dr. Alex James, University of Canterbury.

Introduction

Devaney [1992] describes dynamical systems as “the branch of mathematics that attempts to describe processes in motion”. Dynamical systems are mathematical models of systems that change with time and can be used to model either discrete or continuous processes. Continuous dynamical systems *e.g.* mechanical systems, chemical kinetics, or electric circuits can be modeled by differential equations. Discrete dynamical systems are physical systems that involve discrete time intervals, *e.g.* certain types of population growth, daily fluctuations in the stock market, the spread of cases of infectious diseases, and loans (or deposits) where interest is compounded at fixed intervals. Discrete dynamical systems can be modeled by iterative maps.

This project considers one-dimensional discrete dynamical systems. In the first section, the behaviour and properties of one-dimensional maps are examined using both analytical and graphical methods. Fixed points, periodic orbits, and bifurcations are introduced using examples from linear and non-linear mappings. The logistic equation is used to introduce chaotic behaviour in the next section, and the concept of symbolic dynamics is introduced using the doubling map and the logistic map. The landmark theorem “period three implies chaos” is described. An example of the extension of iterative processes into the domain of fractals is given in final section.

Maps

Mappings or discrete dynamical systems are described by a difference equation or recurrence relation $x_{n+1} = f(x_n)$ and an initial value x_0 . The resulting sequence of iterates $\{x_0, x_1, x_2, \dots, x_n, \dots\}$ is called an *orbit*.

The function $f(x_n)$ may involve one or more parameters, and the initial value x_0 may be varied, so many different orbits are possible for a given mapping function. Graphical and analytical methods can be used to examine the general behaviour of a mapping as the initial value or parameter is varied without having to examine every possible orbit.

The mapping of the sequence of iterates may be plotted as the iterate number n vs the corresponding iterate x_n . This graph of the sequence of iterates should not be confused with the graph of the mapping function $f(x)$ itself. Figure 1 compares the plot of the sequence of iterates of $x_{n+1} = 2x_n(1 - x_n)$ with the graph of the mapping function $f(x) = 2x(1 - x)$.

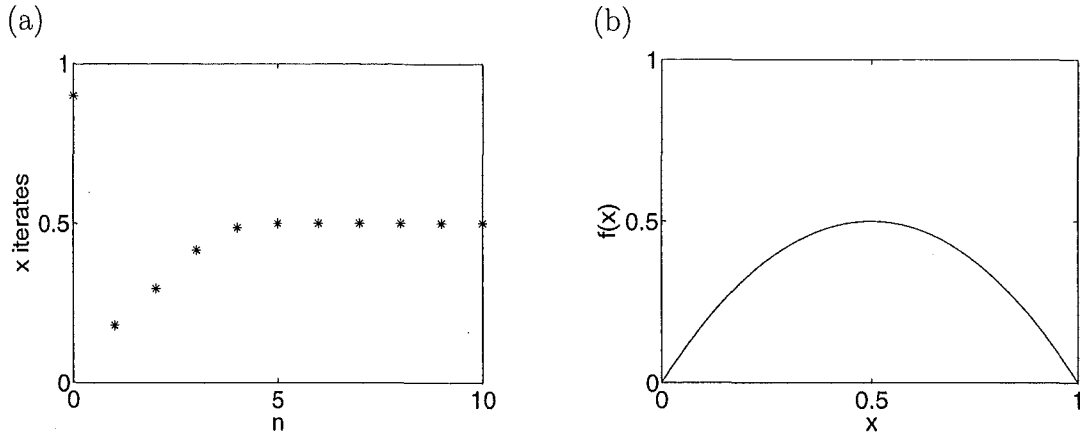


Figure 1: (a) The sequence of iterates for $x_{n+1} = 2x_n(1 - x_n)$ and (b) The mapping function $f(x) = 2x(1 - x)$

Fixed Points and Periodic Orbits

A fixed point of a mapping is a point at which the mapping or orbit remains unchanged for all subsequent iterations. Fixed points may be found analytically by solving the equation $x^* = f(x^*)$. Consider the linear mapping $x_{n+1} = \lambda x_n$ where λ is a parameter. Fixed points of this mapping are given by:

$$x^* = f(x^*) = \lambda x^* \quad \longrightarrow \quad x^* = 0 \quad \text{for } \lambda \neq 0$$

The mapping converges to $x^* = 0$ for $|\lambda| < 1$. For these values of λ the fixed point is often called an “attracting” fixed point or *attractor*. In dynamical systems, an attractor is a set *e.g.* a point, a set of points, or a line, toward which the system evolves or is attracted. A simple graphical way to find the fixed point of a mapping is to plot the mapping function $y = f(x_n)$ and the line $y = x$ on the same graph. Then the fixed points of the mapping are given by the points of intersection of the graphs of $y = f(x_n)$ and $y = x$. For a more formal mathematical definition of a fixed point see Martelli [1999].

A useful tool for analysing maps is a “cobweb diagram”, which plots successive iterations of the mapping and illustrates the general behaviour of the function. Figure 2 demonstrates the method of preparing cobweb diagrams:

- plot the graphs of $y = f(x_n)$ and $y = x$ on the same set of axes
- begin at $x = x_0$ and find $f(x_0)$ (point P_1 in Figure 2) by traveling up from the x-axis to the curve $y = f(x_n)$
- from $f(x_0)$ move horizontally to the line $y = x$ to point P_2
- choose the x-coordinate of this point as the next x-value, x_1

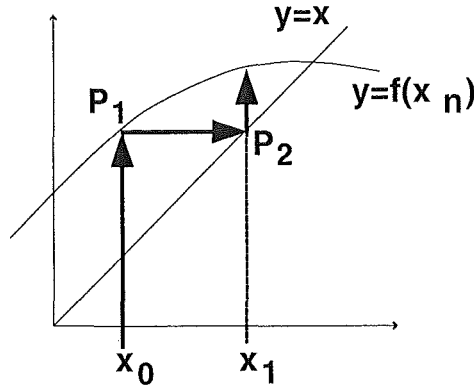


Figure 2: Starting a cobweb diagram: the first iteration

This method works because the y -coordinate of P_1 is $f(x_0)$ as P_1 is on the curve $y = f(x_n)$. Therefore P_2 also has y -coordinate $f(x_0)$ since it is “level” horizontally with P_1 . P_2 also has $f(x_0)$ as its x -coordinate since it is on the line $y = x$, so $f(x_0) = x_1$ as required. If this graphical iteration scheme is repeatedly applied then the sequence of points $\{x_0, x_1, x_2, \dots, x_n, \dots\}$, *i.e.* the orbit of the mapping for the given x_0 , is obtained.

For $0 < \lambda < 1$, the mapping converges monotonically to the fixed point $x^* = 0$. Figure 3 shows the iterative mapping and cobweb diagram for $\lambda = 0.5$. In the cobweb diagram, convergence appears as a “ladder” with $f(x_0)$ at the top of the ladder and “steps” going down until the fixed point $x^* = 0$ is reached.

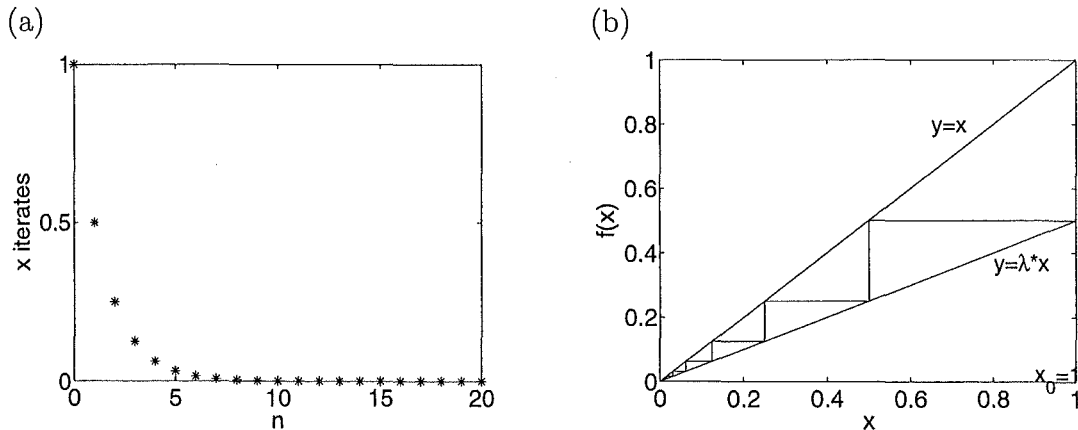


Figure 3: (a) The sequence of iterates for linear mapping with $\lambda = 0.5$ and (b) Cobweb of the linear map with $\lambda = 0.5$ showing convergence to the fixed point $x^* = 0$.

For $-1 < \lambda < 0$, the mapping shows oscillating convergence *i.e.* the iterations approach the fixed point from alternate sides of the fixed point. Figure 4 shows the sequence of iterates and cobweb diagram for $\lambda = -0.5$. The cobweb diagram takes on the spiral shape characteristic of damped oscillations as it approaches the fixed point.

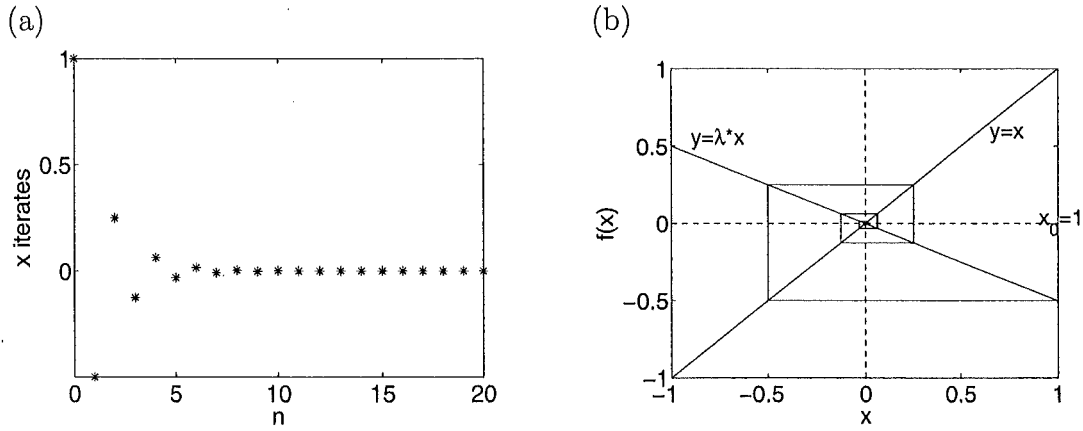


Figure 4: (a) The sequence of iterates for linear mapping with $\lambda = -0.5$ and $x_0 = 1$ and (b) Cobweb of the same linear map showing oscillating convergence to the fixed point $x^* = 0$.

For $\lambda > 1$, the mapping diverges monotonically, as shown by the iterative mapping and also by the cobweb diagram of the function in Figure 5. Again the cobweb shows the divergence as a ladder, but in this case $f(x_0)$ is at the foot of the ladder and the steps go upwards and outwards.

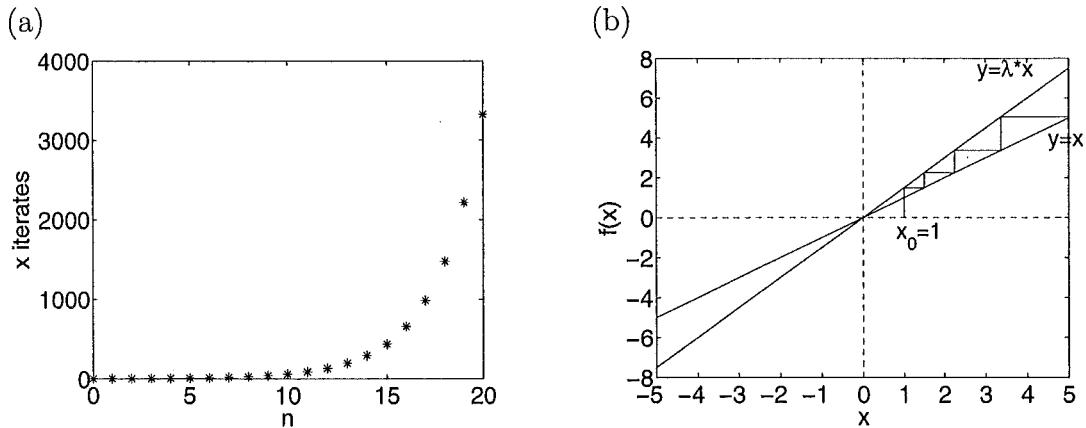


Figure 5: (a) The sequence of iterates for linear mapping with $\lambda = 1.5$ and $x_0 = 1$ and (b) Cobweb of the same linear map showing divergence.

For $\lambda < -1$, the mapping shows oscillating divergence *i.e.* it diverges outwards in an oscillating or alternating fashion, as shown in Figure 6. This time the spiral is moving outward rather than inward toward the fixed point.

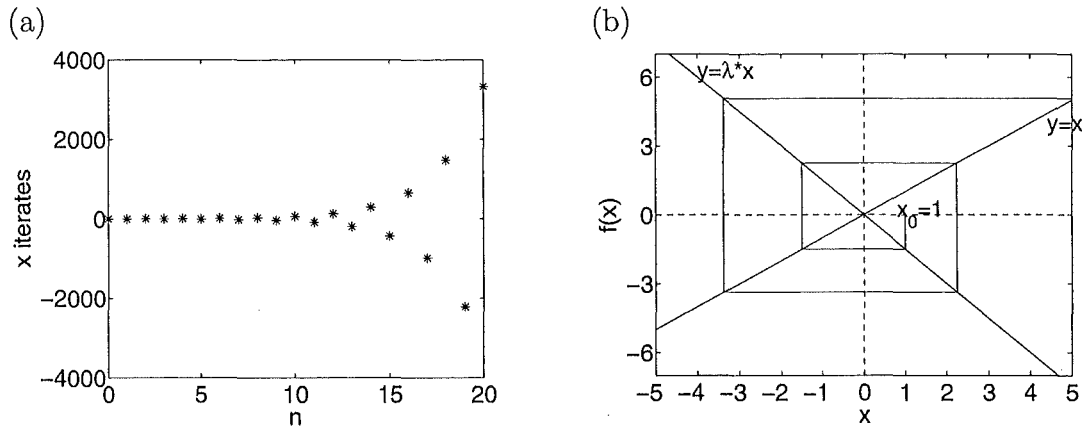


Figure 6: (a) The sequence of iterates for linear mapping with $\lambda = -1.5$ and (b) Cobweb of the linear map with $\lambda = -1.5$ showing oscillating divergence.

From the cobweb diagrams it is clear that if the slope of the line specified by mapping function λx is greater in absolute value than the slope of $y = x$, then the fixed point $x^* = 0$ is repelling. If the slope of the linear map is less in absolute value than the slope of $y = x$, then the fixed point is attracting. This is the basis of the method of linearisation used to determine the stability of fixed points of non-linear maps. For both these cases $|\lambda| \neq 1$, and the fixed point is described as a *hyperbolic* fixed point. For $|\lambda| = 1$ no conclusion can be drawn. In this case the fixed point is described as *non-hyperbolic*.

The linear map $x_{n+1} = \lambda x_n$ illustrates four different types of behaviour of maps: monotonic convergence to a fixed point, oscillating convergence to a fixed point, monotonic divergence to infinity, and oscillating divergence to infinity. Another type of behaviour that can occur in dynamical systems is a periodic orbit.

The linear map $x_{n+1} = ax_n + b$ is called an affine dynamical system, with fixed point $x^* = b/(1 - a)$. The stability of this mapping is also determined by the slope of the linear function, hence for $|a| < 1$ the fixed point is stable, and for $|a| > 1$ the fixed point is unstable. For $|a| = 1$ no conclusion can be drawn, and stability in this case will be considered later. For the case where $a = -1$, the mapping is $x_{n+1} = b - x_n$. The fixed point of this mapping is a function of the parameter, $x^* = b/2$. For all values of x_0 other than the fixed point, this mapping results in a continual oscillation between two values, called a period-2 cycle, see Figure 7. This is an example of a periodic orbit.

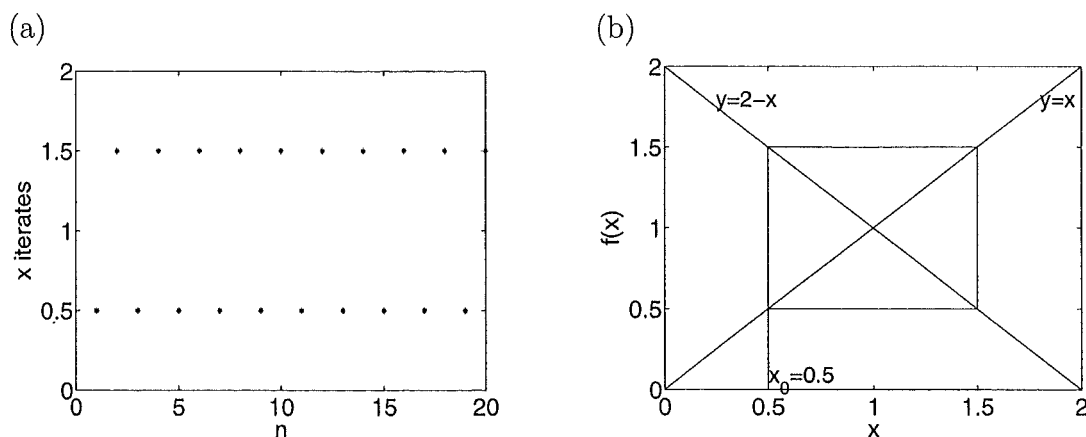


Figure 7: (a) The sequence of iterates for the affine linear mapping $x_{n+1} = 2 - x_n$ and (b) Cobweb of the same linear map. Both graphs illustrate the period-2 cycle.

A mapping has a periodic orbit of period $p > 1$ if $x_{n+p} = x_n$ and $x_{n+i} \neq x_n \forall i < p$. For example, a period-2 orbit occurs when $x = f(f(x)) = f^2(x)$, and a period-3 orbit occurs when $x = f(f(f(x))) = f^3(x)$. A fixed point of the basic mapping f is sometimes denoted as period-1. Points that are fixed points of periodic orbit n but not less than n are sometimes called fixed points of “prime period n ” to distinguish them from fixed points of periods less than n . The existence of a period-3 orbit in a mapping has surprising implications which will be seen later.

Stability of Fixed Points

Fixed points of mappings may be stable, unstable, or semi-stable. A stable fixed point may be compared to the bottom of a well: if an object at the bottom of the well is given a small kick or perturbation it will return there. An unstable fixed point may be compared to the top of a hill: if an object sitting at the top of a hill is given the same kick it will move away and not return. A semi-stable fixed point is similar to a point of inflection. If given a kick from one direction it will return to its place, but if given a kick from the other direction, it will move away and not return. A stable fixed point is also called an attracting fixed point or *attractor* because the system eventually “settles down” to this point. If all starting values x_0 go to this fixed point, then it is a global fixed point, but if only certain x_0 values go to this point then it is a local fixed point.

For a linear mapping, the slope of the linear function determined the stability of the fixed point. To analyse the stability of a fixed point for a non-linear mapping, Figure 8 shows that as the cobweb mapping approaches a fixed point, the cobweb for the tangent to the curve at the fixed point is essentially the same as the cobweb for the curve itself. Thus close to a fixed point the tangent line to the function at that point may be used as an approximation to the mapping function itself.

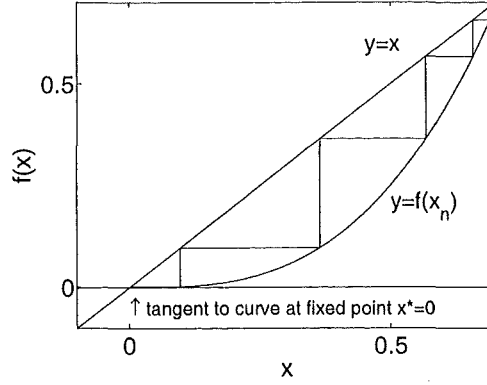


Figure 8: Cobweb diagram for non-linear mapping showing tangent to curve

The equation of the tangent line at $x = x^*$ is given by

$$y - x^* = f'(x^*)(x_n - x^*) \quad (1)$$

$$\implies y = f'(x^*)x_n + x^*(1 - f'(x^*)) \quad (2)$$

which is the form of a linear (affine) mapping with $a = f'(x)$, therefore for:

- $|f'(x^*)| > 1$, x^* is an unstable (repelling) fixed point
- $|f'(x^*)| < 1$, x^* is a stable (attracting) fixed point
- $|f'(x^*)| = 0$, x^* is called a “superstable” fixed point (because it converges very rapidly)

The value $f'(x^*)$, designated λ , is called the *eigenvalue* of the mapping function, and this method of determining the stability of a fixed point is known as *linearisation*. This method can be extended to discrete mappings having more than one dimension, where the constants a and b of the affine mapping are replaced by matrices. Linearisation involves computing the Jacobian evaluated at x^* and testing the eigenvalues of the Jacobian as for the affine case. For further details see Scheinerman [1996].

For the case where $|f'(x)| = 1$, there are two possible approaches to determine the stability of a fixed point. The first is to use a graphical technique such as a cobweb diagram, and the second is to use the following analytical method. If $f'(x^*) = 1$ then the slope of the tangent line at x^* is 1, and therefore the tangent line passes through the point (x^*, x^*) , so the equation of the tangent at x^* must be $y = x$. Then, if $f''(x^*) > 0$ at (x^*, x^*) , the curve of the mapping function is concave up. When the mapping function is concave up, from the cobweb diagram it is evident that x^* is semi-stable from below: stable if $x_0 < x^*$ and unstable if $x_0 > x^*$. Conversely, if $f''(x^*) < 0$ at (x^*, x^*) , the curve is concave down, and it can be seen from the cobweb diagram that x^* is semi-stable from above: stable if $x_0 > x^*$ and unstable if $x_0 < x^*$, shown in Figure 9.

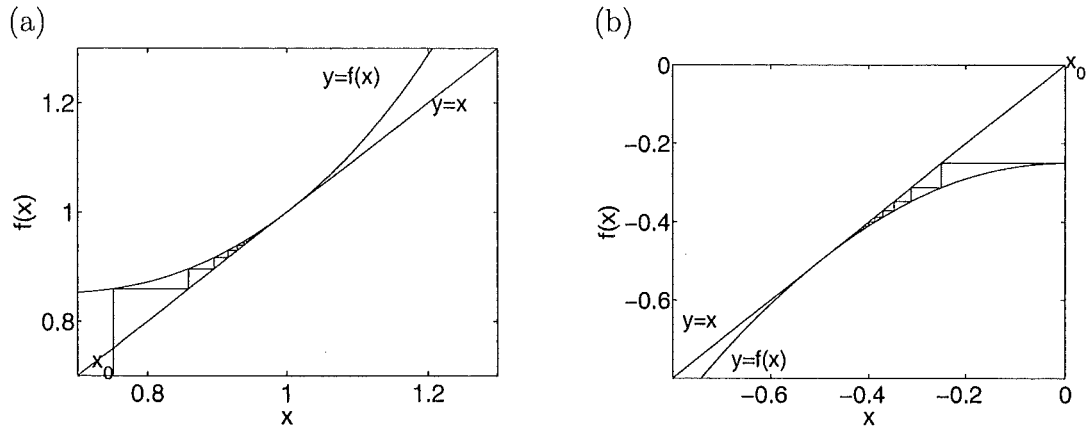


Figure 9: (a) Cobweb for $f''(x) > 0$ showing that the fixed point is semi-stable from below and (b) Cobweb for $f''(x) < 0$ showing that the fixed point is semi-stable from above.

When $f'(x^*) = 1$ and $f''(x^*) = 0$, if $f'''(x) \neq 0$ then x^* is a point of inflection on the mapping function (the concavity of the curve changes at x^*). By examining the concavity of the curve, the stability of the function can be deduced. For $f'''(x^*) < 0$, the fixed point x^* is stable, but for $f'''(x^*) > 0$, x^* is unstable, see Figure 10.

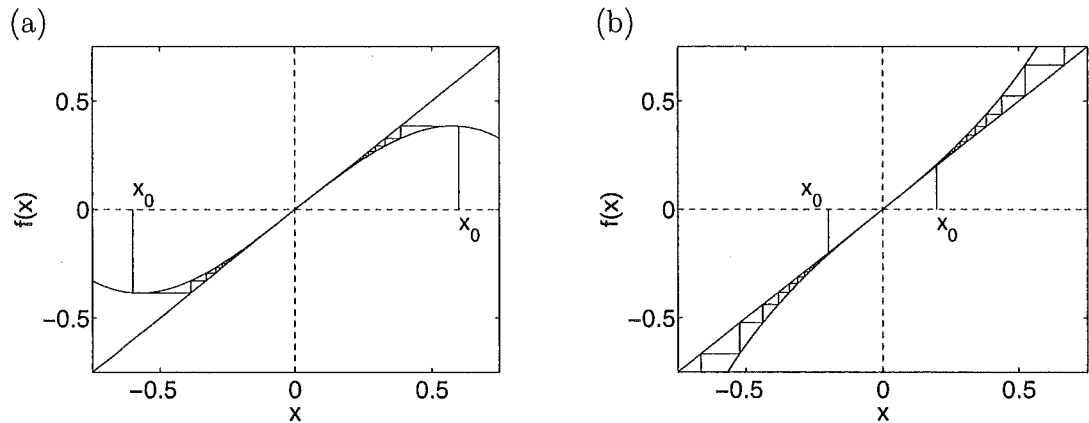


Figure 10: (a) Cobweb for $f'''(x^*) < 0$ showing that the fixed point is stable and (b) Cobweb for $f'''(x^*) > 0$ showing that the fixed point is unstable.

For the case where $f'(x^*) = -1$, the behaviour will be oscillating, which suggests looking at every second point in the iterative mapping, $x_{n+2} = f(f(x_n)) = g(x_n)$. For mapping functions f and g that are both continuous functions, then if x^* is a fixed point of $g(x_n)$ it is also a fixed point of $f(x_n)$, and if x^* is stable for $g(x_n)$ then it is also stable for $f(x_n)$. For further details and proof see Sandefur [1990].

Local Bifurcations

As the parameter of a mapping is varied, the stability of fixed points may change, and fixed points or periodic orbits may appear and disappear. A qualitative change in the behaviour of a mapping as a parameter is varied is called a *local bifurcation*. Bifurcations can only occur at non-hyperbolic fixed points ($|\lambda| = 1$), and they are classified by the way in which fixed points of the mapping change in location, number, and stability. Four different types of bifurcations are possible for one-dimensional discrete mappings: transcritical, pitchfork, saddle-node, and period-doubling. A bifurcation diagram illustrates the change in fixed points as a parameter is varied by plotting the parameter on the horizontal axis against the fixed points or steady states on the vertical axis. Unstable fixed points are shown by a dotted line, and stable fixed points by a solid line.

Transcritical Bifurcations

A *transcritical bifurcation* occurs when two fixed points intersect or “collide” at a specific value of the parameter and their stability interchanges *i.e.* the stable fixed point becomes unstable and *vice versa*. Figure 11(a) gives an example of a transcritical bifurcation using the mapping $x_{n+1} = ax_n - x_n^2$. This mapping has two fixed points, $x^* = 0$ and $x^* = a$. At $a = 0$, the two fixed points coincide, and a bifurcation occurs where the stability of the fixed points changes. For $b < 0$, $x^* = 0$ is stable and $x^* = b$ is unstable. For $b > 0$, $x^* = 0$ is unstable and $x^* = b$ is stable, thus a transcritical bifurcation has occurred.

A *pitchfork bifurcation* occurs when a fixed point becomes changes its stability and “splits” into three fixed points as the parameter is varied. The original fixed point is bracketed by the two new fixed points. A supercritical pitchfork bifurcation has a fixed point which changes from stable to unstable with two stable fixed points appearing on either side, shown in Figure 11(b), while a subcritical pitchfork bifurcation has a fixed point which changes from unstable to stable with two unstable fixed points appearing on either side.

The *saddle-node bifurcation*, or fold bifurcation, occurs when two fixed points move toward each other, intersect and then both disappear; alternately this can be thought of as two fixed points appearing as parameter is varied. The mapping $x_{n+1} = \mu - (x_n)^2$ undergoes a bifurcation when $\mu = -0.25$. When $\mu < -0.25$, the mapping has no fixed points. At $\mu = -0.25$, the fixed point $x^* = -0.5$ appears. When $\mu > -0.25$, the mapping has two fixed points

$$x^* = \frac{-1 \pm \sqrt{1 + 4\mu}}{2}$$

one of which is stable and the other unstable, as shown in Figure 12(a).

The final bifurcation seen in discrete mappings is the *flip bifurcation* or period-doubling bifurcation. The importance of this bifurcation will be seen in the next section. In a period-doubling bifurcation, a fixed point changes its stability and a period-2 cycle which “brackets”

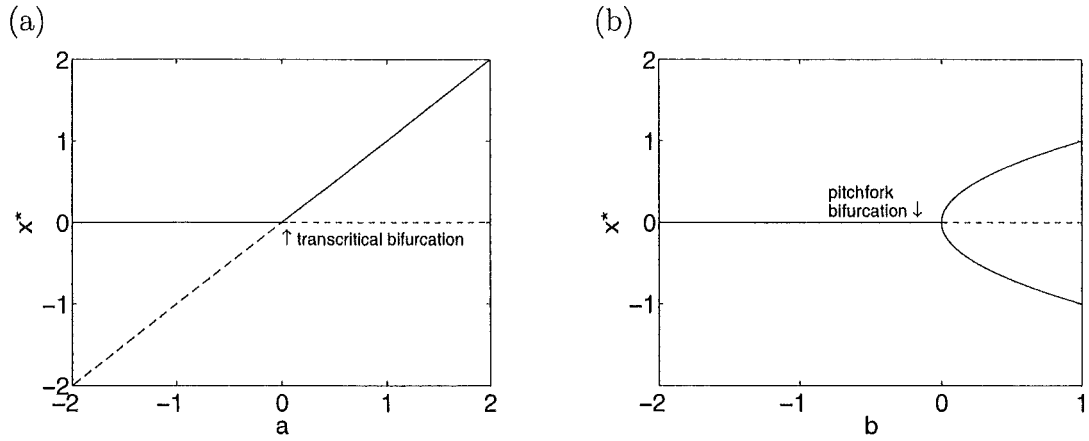


Figure 11: (a) Transcritical bifurcation example $x_{n+1} = ax_n - x_n^2$ (b) Supercritical pitchfork bifurcation example $x_{n+1} = bx_n - (x_n)^3$

the original fixed point begins. If the fixed point changes from stable to unstable, the period-2 orbit which emerges is stable, called a supercritical flip bifurcation. If the fixed point changes from unstable to stable, then the period-2 orbit that emerges is unstable, called a subcritical flip bifurcation. Figure 12(b) shows an example of a supercritical flip bifurcation. The flip bifurcation occurs when $\lambda = -1$. Note the difference between this bifurcation and the pitchfork bifurcation in Figure 11(b). In the pitchfork bifurcation two new (period-1) fixed points appear, while in a flip bifurcation a new period-2 orbit appears. Transcritical, saddle node, and pitchfork bifurcations all occur in continuous time dynamical systems also, but the flip bifurcation is unique to discrete systems.

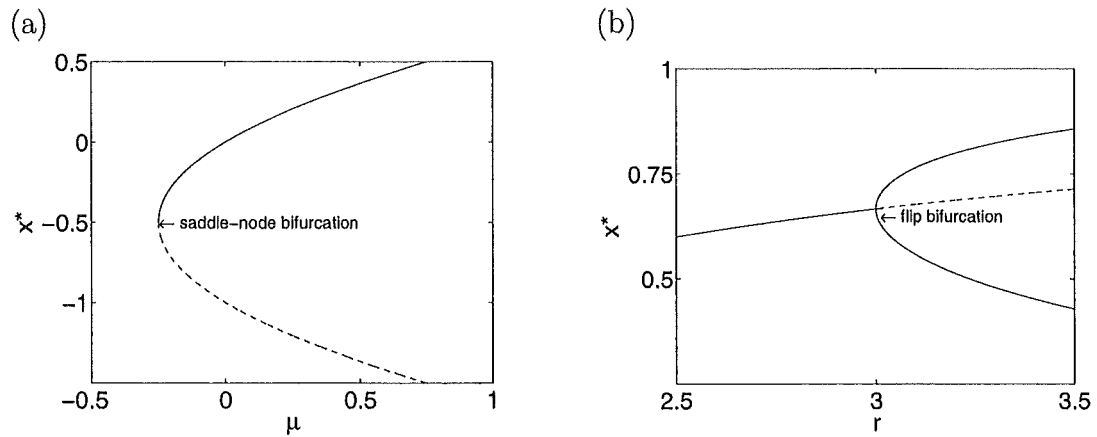


Figure 12: (a) Saddle-node bifurcation example $x_{n+1} = \mu - x_n^2$ (b) Flip or period-doubling bifurcation example $x_{n+1} = rx_n(1 - x_n)$

Chaos

In the early 1970's Robert May [May, 1976] was studying population growth using a discrete time model, $x_{n+1} = rx_n(1 - x_n)$, called the *logistic equation*. The parameter r represents the growth rate and x_n is the population proportion at generation n . The $(1 - x_n)$ term inhibits population growth, modeling limiting factors, because as x approaches 1, $(1 - x_n)$ approaches 0. For initial values x_0 outside the interval $[0, 1]$ the discrete mapping very quickly diverges to negative infinity. However for initial values in the interval $[0, 1]$, May observed some curious behaviour as the parameter r varies from $r = 0$ to $r = 4$.

The fixed points of the mapping are $x^* = 0$ and $x^* = (r - 1)/r$. For $0 \leq r < 1$, $x^* = 0$ is a stable fixed point or attractor, and the population goes to extinction. For $1 < r < 3$, $x^* = (r - 1)/r$ is the attractor, and the population will eventually settle at this non-zero steady state. Therefore a transcritical bifurcation occurs at $r = 1$, as the two fixed points intersect at $x^* = 0$ and their stability changes. Note that $f'(x) = r - 2rx$ and $f'(x^*) = f'(0) = 1$ for $r = 1$, so $x^* = 0$ is a non-hyperbolic fixed point at $r = 1$.

The next bifurcation occurs at $r = 3$, when $x^* = (r - 1)/r = 2/3$. Then $f'(x^*) = f'(2/3) = -1$, and $x^* = 2/3$ is a non-hyperbolic fixed point. A period-doubling bifurcation occurs, with a stable period-2 cycle or two-point attractor emerging. The fixed points of the period-2 cycle are found by solving $f^2(x) = f(f(x))$ for the logistic equation, giving

$$p_1, p_2 = \frac{(r + 1) \pm \sqrt{(r - 3)(r + 1)}}{2r}$$

and these fixed points are stable for $3 < r < 1 + \sqrt{6}$.

For $3 < r < 4$ some very strange things start happening. The algebra gets very messy from this point onwards, so a numerical approach will be taken using Matlab. A bifurcation diagram is plotted by iterating the mapping function many times for each r value so the system has “settled” down to the attractor, discarding the early iterations, and plotting the remaining orbit against the r value. This type of diagram only shows the stable periodic orbits. Figure 13(a) shows a series of period-doubling bifurcations, with the period-2 attractor being replaced by a period-4 attractor, then a period-8 attractor, *etc.* These period-doubling bifurcations come at closer and closer intervals until a critical point is reached where there are an infinite number of periodic orbits, and the map becomes “chaotic”, or densely filled with points from unpredictable orbits, shown in Figure 13(b).

The chaotic region is interspersed with periodic “windows” as seen in Figure 14(a). A periodic window occurs when the mapping makes a transition from chaotic behaviour to an attracting n -cycle. Each branch of the n -cycle in these windows then gives rise to a new period-doubling transition to chaos. If the periodic window is magnified near one branch then a diagram similar to the whole diagram appears, called self-similarity, see Figure 14(b).

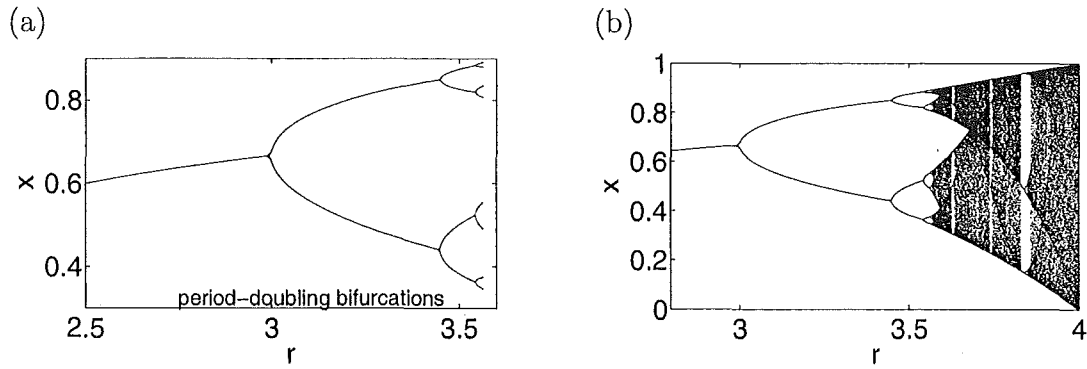


Figure 13: (a) Logistic bifurcation diagram for $2.5 < r < 3.54$ showing period-doubling transitions (b) Logistic bifurcation diagram for $2.8 < r < 4$ showing chaotic behaviour

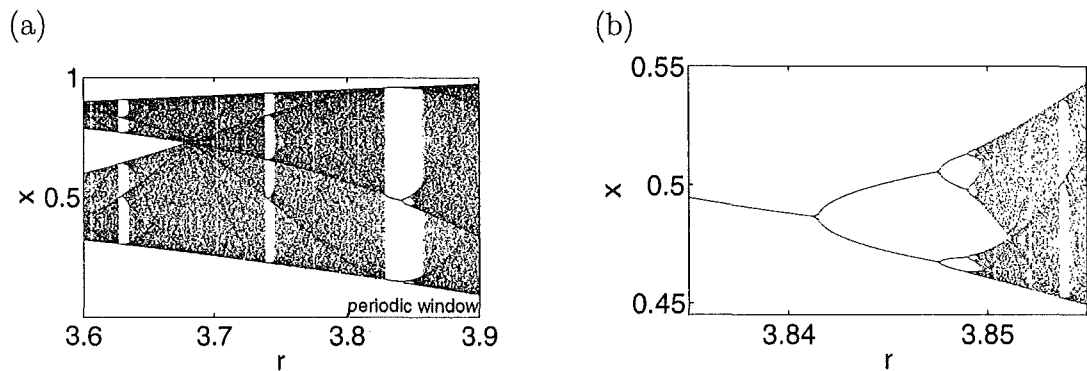


Figure 14: (a) Logistic bifurcation diagram for $3.6 < r < 3.9$ showing periodic windows in chaotic region and (b) magnification of region $3.835 < r < 3.855$ showing a portion of the periodic window and self-similar behaviour

The logistic map is a perfectly deterministic system, with a specified equation and initial condition, and yet it is not predictable but shows chaotic behaviour. Exactly what is “chaos”? There is no precise mathematical definition of chaos, and descriptions usually consist of a set of properties that describe a chaotic dynamical system:

1. the system has periodic orbits of all possible periods, and the periodic orbits are dense in the chaotic region
2. the system has a point whose orbit is dense *i.e.* an orbit can be found which goes arbitrarily close to any given point in the chaotic region
3. the system shows sensitive dependence on initial conditions

Chaos occurs in simple one-dimensional non-linear maps as well as more complex systems, and in physical systems such as the weather, electronic circuits, complex chemical reactions, and biological systems *e.g.* heartbeats, metabolic networks.

Symbolic Dynamics

Because a precise mathematical definition of “chaos” has yet to emerge, proving that a dynamical system is chaotic is difficult: how do you prove that a system has a given set of properties at all times? One means is by the use of symbolic dynamics.

Symbolic Dynamics for the Doubling Map

The doubling map or bakers map is described by $x_{n+1} = 2x \pmod{1}$ or piecewise by:

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2 \\ 2x - 1 & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

The doubling map can be divided in half, with the interval $[0, 0.5)$ represented by the symbol 0 and the interval $[0.5, 1]$ by the symbol 1, as in Figure 15.

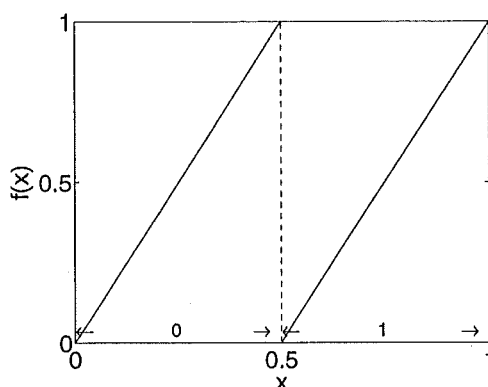


Figure 15: Symbolic dynamics for the Doubling Map

The orbit of a point $x_0 \in [0, 1]$ can be described as the sequence of 0's and 1's given by which interval contains each iteration, that is, the order in which the orbit visits 0 or 1. For example, consider the orbit of $x_0 = 0.6$:

0.6 0.2 0.4 0.8 0.6 0.2 0.4 0.8 0.6 0.2 0.4 0.8...

The sequence of 0's and 1's for this orbit is the infinite binary sequence 0.100110011001,... or $0.\overline{1001}$. This sequence is actually the binary representation of 0.6 because $2x \pmod{1}$ is the procedure for converting a decimal fraction to a binary fraction. This representation of an orbit by a symbolic sequence is the principle of symbolic dynamics. For the doubling map, a binary sequence is used to develop symbolic dynamics. All the points in $[0, 1]$ can be represented in binary form, and this binary sequence is exactly the orbit of the point under the doubling map.

The behaviour of the mapping can be studied in terms of its effect on the binary sequence. Each iteration of the mapping involves a shift in the binary sequence. If the first digit after

the decimal point is 0, our initial value is in the first interval, and the iteration simply means discarding this first digit *i.e.* shifting the sequence one place to the left. If the first digit after the decimal point is 1, our initial value is in the second interval, and an iteration involves shifting the sequence one place to the left and discarding the left most digit of 1:

$$0.a_0a_1a_2a_3\dots \longrightarrow 0.a_1a_2a_3a_4 \quad \text{where the shift "forgets" } a_0$$

Note also that the graph of the doubling map function can be repeatedly divided into equal subintervals described by a symbolic binary sequence. These sequences specify the start of the orbits of all the points in that interval. The two halves are divided into four subintervals, 00, 01, 10, 11. The next 8 subintervals are 000, 001, 010, 011, 100, 101, 110, and 111. This can be continued to give n segments each of length $\frac{1}{2^n}$.

The fixed point of the map is $x^* = 0$. $x_0 = 1$ maps to 0 and then remains there, so the mapping can start at 1 but no other initial value maps to 1. The binary sequence 0.11111... or $0.\bar{1}$ is regarded as the same as 1 so finite representations are used in place of sequences that end in an infinite series of 1's. Periodic orbits are easy to find since any repeating binary sequence is periodic, *e.g.* $x_0 = 0.\overline{01}$ results in a period-2 orbit, $x_0 = 0.\overline{001}$ a period-3 orbit, and $x_0 = 0.\overline{a_1a_2a_3\dots a_n}$ a period- n orbit. All rational numbers are represented as repeating binary sequences, and hence all rational numbers eventually have periodic orbits. Because there are an infinite number of rational numbers in the interval $[0, 1]$, periodic orbits of all possible periods are present. In fact there are 2^n periodic points of period- n (but not necessarily *prime* period- n).

Next consider a point whose orbit is *dense* in the chaotic region. This orbit is not periodic, but comes arbitrarily close to any designated point in the region. To show that such an orbit exists, consider the point:

$$p = [\underbrace{0\ 0}_{1\text{-blocks}} \underbrace{00\ 01\ 10\ 11}_{2\text{-blocks}} \underbrace{000\ 001\ 010\ 011\ 100\ 101\ 110\ 111}_{3\text{-blocks}} \underbrace{\dots}_{4\text{-blocks}} \dots]$$

which is the sequence consisting of all possible finite blocks of 0's and 1's. As this sequence is iterated and drops the leading digit each time, every possible subsequence of any given length will appear at the beginning of the sequence. This orbit contains points which approximate to any desired accuracy every point in the interval $[0, 1]$. Therefore the doubling map has a dense orbit, something like an infinite thread which stitches the region together in a random "web".

Finally consider sensitive dependence on initial conditions. This means that two arbitrarily close initial conditions will diverge and behave independently after a finite number of mappings, even though their orbits are still bounded in the mapping region. Suppose x_0 and y_0 are very close together. Then the first, say n , entries in their symbolic sequence will be the same:

$$x_0 = [\underbrace{0.01100\dots\dots 110\ 010\dots\dots}] \quad \text{and} \quad y_0 = [\underbrace{0.01100\dots\dots 110\ 101\dots\dots}]$$

n entries n entries

After each iteration, the first binary digit after the decimal point is removed in the shift, and some information is lost. Then to be able to predict the orbit of some x_0 completely, x_0 must be known to infinite accuracy. Even if a very large number of digits are known, say the first N digits, it still cannot be predicted whether the orbit lies above or below $1/2$ at the $(N+1)$ -th iteration. If two initial conditions differ only after N digits, so that they lie within a distance $1/(2^N - 1)$ of each other, after the first N iterations they will diverge to opposite sides of $1/2$ and thereafter continue to behave independently. Hence there is sensitive dependence on initial conditions. This is the principle behind the popular image of chaos: A butterfly fluttering its wings may cause a thunderstorm on the other side of the world!

Symbolic Dynamics for the Logistic Map

This analysis is drawn from the symbolic dynamics for the doubling map and from a similar analysis of the quadratic mapping $x_{n+1} = x_n^2 - 2.64$ found in Scheinerman [1996]. Consider the logistic map with $r = 4$. Divide this map in half, and let L be the region $[0, 1/2)$ and R the region $[1/2, 1]$, as shown in Figure 16. Then divide these regions again according to the region each point maps to under the logistic map. These regions will not be of equal length as they were for the doubling map, since the logistic map is a non-linear map. The first four subintervals are as follows:

$$\begin{array}{ll}
 LL & [0, \frac{1}{2} - \frac{1}{(2\sqrt{2})}) \\
 LR & [\frac{1}{2} - \frac{1}{(2\sqrt{2})}, \frac{1}{2}) \\
 RR & [\frac{1}{2}, \frac{1}{2} + \frac{1}{(2\sqrt{2})}) \\
 RL & [\frac{1}{2} + \frac{1}{(2\sqrt{2})}, 1].
 \end{array}$$

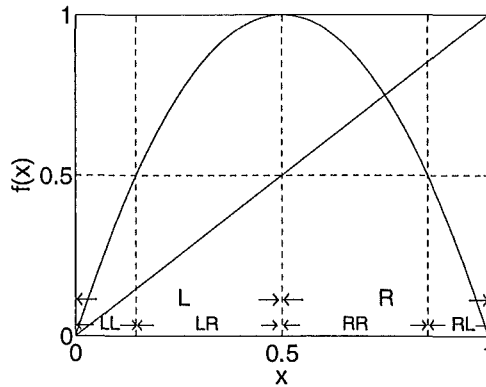


Figure 16: Symbolic dynamics for the Logistic Map with $r=4$

The next division results in 8 subregions as follows: LLL , LLR , LRR , LRL , RRL , RRR , RLR , RLL . The endpoints of these interval are found by solving the equation

$$4x(1-x) = \frac{1}{2} \pm \frac{1}{(2\sqrt{2})}.$$

These divisions can be continued infinitely, giving smaller and smaller subregions of the interval $[0, 1]$. Any point in this interval is uniquely described by an infinite sequence of R 's and L 's. For example, $x_0 = 1/4$ starts in the region L , maps to $x_1 = 3/4$ which is in the region R , then continues mapping to $x_n = 3/4$ in the region R for all $n \geq 1$ since this is a fixed point. Therefore its symbolic representation is $[LRRRRRR.....]$ where there are an infinite number of R 's. Similarly, $x_0 = 1/2$ has the representation $[LRLLLLL.....]$. If x_0 has the representation $[LRLRR....]$ then x_1 has the representation $[RLRRR....]$, i.e. to iterate the map simply drop off the first symbol in the sequence. The mapping then becomes a shift in the symbolic representation, as for the doubling map.

The fixed points of the map are $x^* = 0$ with sequence $[LLLL....] = [\bar{L}]$ and $x^* = 3/4$ with sequence $[RRRR....] = [\bar{R}]$. The symbolic dynamics make it possible to show that there are periodic orbits of all periods present in this map. If $x_0 = [\bar{RL}]$ then $x_1 = [\bar{LR}]$ and $x_2 = [\bar{RL}]$ and so $[\bar{RL}]$ and $[\bar{LR}]$ give a period-2 orbit of the map. There are two period-3 orbits, made up of sequences $[\bar{LRL}]$, $[\bar{RLL}]$, $[\bar{LLR}]$ and $[\bar{RLR}]$, $[\bar{LRR}]$, $[\bar{RRL}]$. This can be extended to an infinite number of sequences with all possible sequences of L and R , so orbits of all periods occur, therefore the map has periodic orbits of all possible periods.

The existence of a dense orbit is shown by considering the symbolic sequence:

$$p = [\underbrace{R L}_{1\text{-blocks}} \underbrace{LL LR RL RR}_{2\text{-blocks}} \underbrace{LLL LLR LRR LRL RRL RRR RLR RLL}_{3\text{-blocks}} \underbrace{\dots}_{4\text{-blocks}} \dots]$$

which is the sequence consisting of all possible blocks of R 's and L 's of length 1, followed by all such possible blocks of length 2, then length 3, and so on. This point p has an orbit that is dense, since for any point x that is chosen, p can be iterated, shifting the map until the sequence is reached that matches the chosen point within the specified tolerance.

The argument for sensitive dependence on initial conditions is similar to that for the doubling map. After each iteration, the first letter in the symbolic sequence is dropped, and some information is lost. Suppose x_0 and y_0 are very close together. Then the first, say n , entries in their symbolic sequence will be the same:

$$x_0 = [\underbrace{LRLL.....LLR}_{n \text{ entries}} LRL.....] \quad \text{and} \quad y_0 = [\underbrace{LRLL.....LLR}_{n \text{ entries}} RLR.....]$$

When the mapping is iterated, for the first n iterations the points will remain close together, but after the n^{th} iteration the points will diverge. To predict the complete orbit of an initial point would require knowledge of the infinite symbol sequence, and this is not possible.

Universality in Unimodal Maps

Consider the period-doubling transitions or “cascade to chaos” seen in the logistic map. These transitions occur at the following values [Strogatz, 1994]:

$r_1 = 3$	period-2 cycle
$r_2 = 3.449\dots$	period-4 cycle
$r_3 = 3.54409\dots$	period-8 cycle
$r_4 = 3.564407\dots$	period-16 cycle
$r_5 = 3.568759\dots$	period-32 cycle
\dots	
\dots	
$r_\infty = 3.569946\dots$	infinite number of periodic cycles

Mitchell Feigenbaum studied the intervals between successive period-doubling transitions in the logistic map and found that they get smaller and smaller at the same rate: if $\Delta_n = r_n - r_{n-1}$ is the distance between two successive bifurcation points and $\Delta_{n+1} = r_{n+1} - r_n$ is the distance between the next two bifurcation points, then

$$\frac{\Delta_n}{\Delta_{n+1}} \longrightarrow \delta \quad \text{as} \quad n \longrightarrow \infty$$

where $\delta = 4.6692\dots$ is now known as Feigenbaum’s number.

Amazingly, Feigenbaum found that this number is universal for unimodal maps: they have the same period doubling behaviour with exactly the same constant δ . Unimodal mappings include not only polynomial functions, but also transcendental functions *e.g.* $x_{n+1} = \mu \sin(\pi x_n)$, and piecewise functions *e.g.* the tent map. Unimodal maps are smooth, concave down, have a single maximum on the interval $x \in [0, 1]$ with $f(0) = f(1) = 0$. Such maps follow exactly the same sequence of periodic cycles and chaos, called the U-sequence or universal sequence [Strogatz, 1994], and have the same convergence rate for r_n as $n \rightarrow \infty$. Feigenbaum’s constant is as fundamental to period-doubling in unimodal maps as the constant π is to circles.

Another universal constant associated with the period-doubling cascade to chaos is obtained by finding the ratio of the distances between the “arms” of each period 2 cycle at the subsequent bifurcation point. For a more precise definition see Strogatz [1994] p. 373. This ratio tends to the following limit as $n \rightarrow \infty$:

$$\frac{d_n}{d_{n+1}} \longrightarrow \alpha = -2.5029\dots$$

Feigenbaum [Feigenbaum, 1979] then used the principle of renormalization from statistical physics to explain why δ and α are universal constants for unimodal maps.

Period Three Implies Chaos

The first mention of the word “chaos” in scientific literature occurred in 1975, in a paper by Li and Yorke with the title “Period three implies chaos” [Li and Yorke, 1975]. However this theorem is actually a special case of an earlier theorem, Sarkovskii’s theorem, which was originally published in Russian in 1964 and unknown in the west until Li and Yorke published their work. An informal outline of this proof follows, based on the proof described in Scheinerman [1996].

The Period three theorem states that for a mapping $x_{n+1} = f(x_n)$ where $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then if f has a periodic orbit of period-3, then f also has periodic orbits of *all* other periods. The first step is to show that if f has a periodic orbit of period-2, then f must have a fixed point. If a period-2 orbit exists, there are two points a and b such that $a \neq b$, $f(a) = b$ and $f(b) = a$ for the mapping $x_{n+1} = f(x_n)$. Hence (a, b) and (b, a) are two points on the graph of the function.

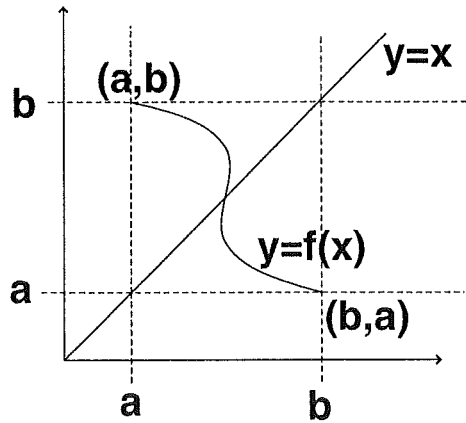


Figure 17: Period-2 implies a fixed point

(a, b) and (b, a) must lie on opposite sides of the line $y = x$, as shown in Figure 17. Because f is a continuous function, the curve $y = f(x)$ must cross the line $y = x$ between $x = a$ and $x = b$, and this point of intersection must be a fixed point of f . Therefore if f has a period-2 orbit, it must also have a fixed point (period-1).

Next consider the case where f has a period-3 orbit. Then there exist three distinct points $a < b < c$ (without loss of generality) for which $a \neq b \neq c$ and

$$f(a) = b \quad f(b) = c \quad f(c) = a$$

Hence the points (a, b) , (b, c) , and (c, a) are points on the graph of $f(x)$. Figure 18(a) shows that since (a, b) is above the line $y = x$ and (c, a) is below $y = x$, and the curve $y = f(x)$ is continuous, it must cross $y = x$ between $x = b$ and $x = c$. Therefore the mapping must have a fixed point.

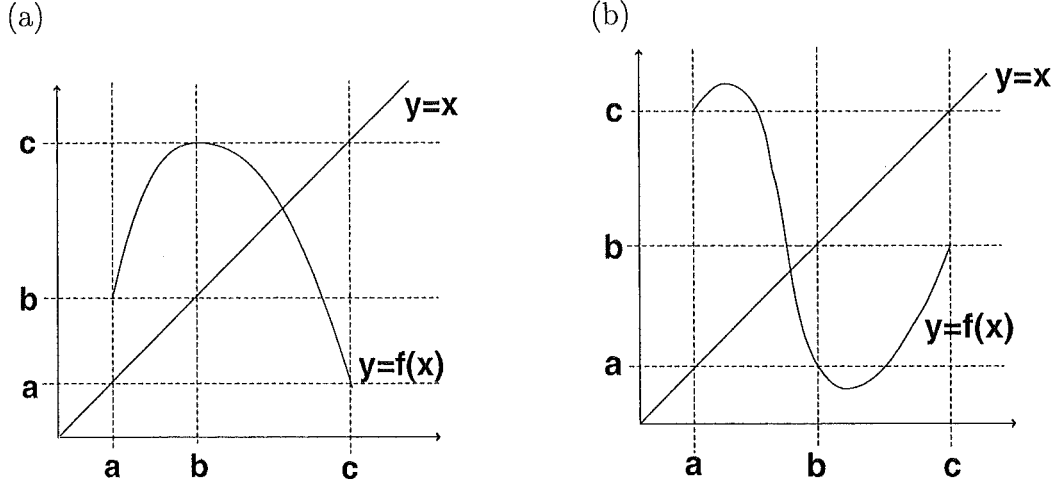


Figure 18: (a) Period-3 implies a fixed point and (b) Period-3 implies Period-2

To show that a period-3 orbit implies a period-2 orbit, consider $f(f(x)) = f^2(x)$. A period-2 orbit must satisfy $x = f(f(x)) = f^2(x)$. Since f has a period-3 orbit:

$$\begin{aligned} f^2(a) &= f(f(a)) = f(b) = c \\ f^2(b) &= f(f(b)) = f(c) = a \\ f^2(c) &= f(f(c)) = f(a) = b \end{aligned}$$

Therefore the graph of $f^2(x)$ must go through the points (a, c) , (b, a) , and (c, b) . The points (a, b) and (b, a) are on opposite sides of the line $y = x$, as shown in Figure 18(b), so the curve $y = f^2(x)$ must cross $y = x$ between $x = a$ and $x = b$. Therefore the mapping of $f^2(x)$ must have a fixed point. But this is only a period-2 orbit **if** the fixed point is not also a fixed point of $f(x)$. To check this, let d be the greatest (last) fixed point of f in the interval $[a, b]$. Then $f(d) = d$ and for $d < x \leq b$ there are no fixed points of f , *i.e.* $x \neq f(x)$ in $(d, b]$. Thus the curve $y = f(x)$ must go from the point (d, d) to the point (b, c) without crossing the line $y = x$. Since $d < b < c$ and f is continuous, the graph of f must cross the line $y = b$ between $x = d$ and $x = b$. Let p be the point where the graph of f crosses $y = b$, so that $f(p) = b$ and $d < p < b$, shown in Figure 19. Now consider the graph of $y = f^2(x)$ at point p :

$$f^2(p) = f(f(p)) = f(b) = c$$

so $f^2(x)$ goes through (p, c) , which is on the opposite side of the line $y = x$ to (b, a) . Hence in the interval (p, b) there exists a point where $f^2(x) = x$, and this point cannot be a fixed point of f since $p > d \implies x > d$ and d was assumed to be the greatest fixed point of f in the interval (a, b) . Therefore x is a fixed point of $f^2(x)$ only and not of $f(x)$, and f has a period-2 orbit.

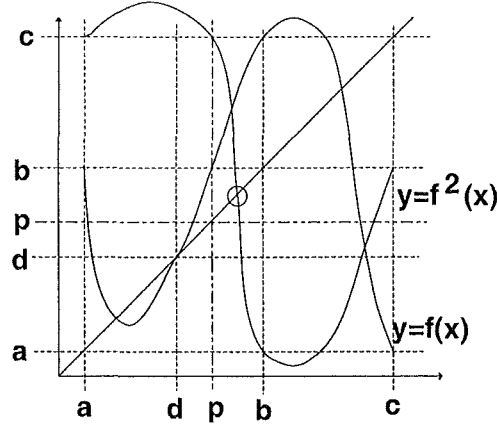


Figure 19: Fixed point of $f^2(x)$ is prime period-2

Next it must be shown that a period-3 orbit implies a period-4 orbit. Consider a closed interval $I = [p, q]$ so that $f(I)$ is the set of all values $f(x)$ where $x \in I$.

Statement 1: If $f(I) \supseteq I$ then $f(x) = x$ for some $x \in I$.

$f(I)$ is a superset of I , i.e. $f(I)$ ranges over all the values contained in I . Figure 20(a) shows this graphically: if the function values contain all the input values or domain values, $[p, q] \in f(I)$, then the function must cross the line $y = x$ as it ranges from $f(I) = q$ to $f(I) = p$. This is a consequence of the Intermediate Value theorem. Therefore there must be a fixed point $x^* = f(x^*)$ in the interval $I = [p, q]$.

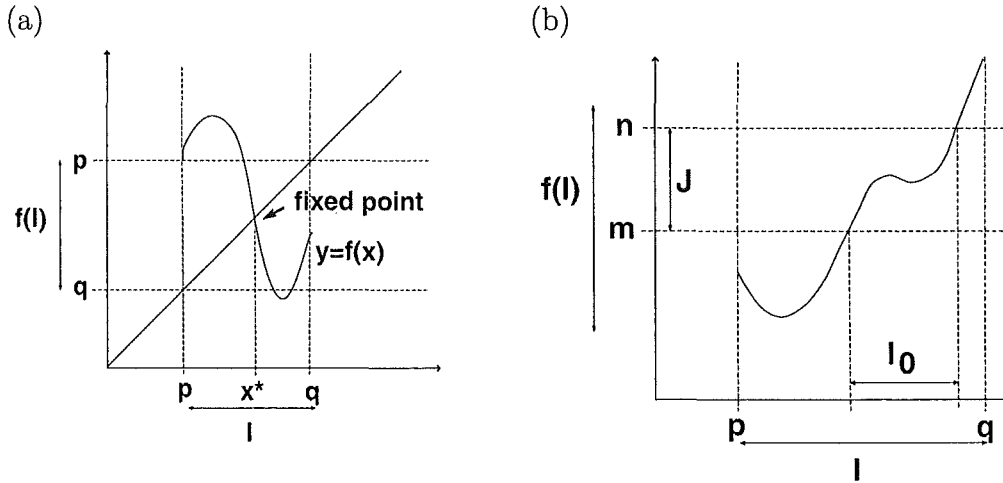


Figure 20: (a) If $f(I) \supseteq I$ then $f(x) = x$ for some $x \in I$ and (b) If $f(I) \supseteq J$, I contains a closed subinterval I_0 for which $f(I_0) = J$.

Next consider two closed intervals $I = [p, q]$ and $J = [m, n]$ with $f(I)$ and $f(J)$ defined as before.

Statement 2: If $f(I) \supseteq J$, I contains a closed subinterval I_0 for which $f(I_0) = J$.

This is illustrated in Figure 20(b). Let $I_0 = [b, c]$. Since $f(b) = c$ and $f(c) = a$ then $f(I_0) \supseteq [a, c] \supseteq [b, c] = I_0$ thus $f(I_0) \supseteq I_0$ and so Statement 2 can be applied to choose $I_1 \subseteq I_0$ so that $f(I) = I_0$. Then since $f(I) = I_0$ and $I_0 \supseteq I_1$, $f(I_1) \supseteq I_1$ and Statement 2 can be applied again to choose $I_2 \subseteq I_1$ such that $f(I_2) = I_1$. Now:

$$\begin{aligned} f^3(I_2) &= f^2(f(I_2)) \\ &= f^2(I_1) \\ &= f(f(I_1)) \\ &= f(I_0) \\ &\supseteq [a, c] \\ &\supseteq [a, b] \end{aligned}$$

and so, using Statement 2 yet again to choose $I_3 \subseteq I_2$ so that $f^3(I_3) = [a, b]$ which gives:

$$\begin{aligned} f^4(I_3) &= f[f^3(I_3)] \\ &= f([a, b]) \\ &\supseteq [b, c] && \text{because } f(a) = b \text{ and } f(b) = c \\ &= I_0 && \text{because } I_0 = [b, c] \\ &\supseteq I_1 \\ &\supseteq I_2 \\ &\supseteq I_3 \end{aligned}$$

and therefore $f^4(I_3) \supseteq I_3$ and by Statement 1 f^4 has a fixed point in I_3 , that is, $x \in I_3$ and $x = f^4(x)$.

To show that x is a fixed point of period-4 only, *i.e.* prime period-4, and not of any other period, note that $x \in I_3 \subseteq I_0 = [b, c]$ so $x \in [b, c]$. Since $f^3(I_3) = [a, b]$ and $x \in I_3$ then $f^3(x) \in [a, b]$.

Also

$$\begin{aligned}
I_3 &\subseteq I_2 &\Rightarrow & x \in I_2 \\
f(I_2) &= I_1 &\Rightarrow & f(x) \in I_1 \\
f^2(x) &= f(f(x)) &\text{and} & f(I_1) = I_0 \\
\Rightarrow f(f(x)) \in I_0 &= [b, c]
\end{aligned}$$

In summary, for the fixed point x found for period-4, $x \in I_2$, $f(x) \in I_1$ and $f^2(x) \in I_0$. Thus all $x, f(x)$, and $f^2(x)$ are in the interval $[b, c]$ and $f^3(x) \in [a, b]$. Now suppose that x is periodic with period less than 4, then either $x = f(x)$ or $x = f^2(x)$ or $x = f^3(x)$. Since all $x, f(x)$, and $f^2(x)$ are in the interval $[b, c]$ and at most x is period-3, then all iterates $f^k(x) \in [b, c]$ for x period-3 or less. However $f^3(x) \in [a, b]$. Therefore $f^3(x) = b$. But this gives $x = f^4(x) = f(f^3(x)) = f(b) = c$ thus $f(x) = f(c) = a$. However $f(x) \in [b, c]$ and $a \notin [b, c]$ which is a contradiction. Therefore x is a fixed point of f^4 only, and f has a period-4 orbit.

By extending this argument using more subintervals, it can be shown that f has periodic orbits of $n = 5, 6, 7, \dots$. Thus if f has a periodic orbit of period-3, then it also has periodic orbits of all other periods *i.e.* period three implies chaos.

Fractals

Attractors are cycles that a mapping converges to after transients have died away, and they can be fixed points, periodic orbits, or “strange attractors”. Dynamical systems can have many coexisting attractors. A “basin of attraction” for an attractor is the set of initial conditions whose orbits are asymptotic to a given attractor. Strange attractors have a peculiar property: orbits on the attractor diverge from neighbouring orbits and yet remain confined to a bounded region, called the phase space of the mapping. This is achieved by a kind of “stretching and folding” process in the mapping, which creates a self-similar or **fractal** structure in the attractor. This was first observed by Hénon in the discrete two-dimensional Hénon map [Hénon, 1976].

Fractals are geometric shapes with fine structure at arbitrarily small scales and with a degree of self-similarity. If a very small portion of a fractal is magnified, a picture that looks very similar to the whole fractal is seen. This self-similarity is exact for simple fractals such as the Cantor Set and the Koch snowflake. Fractals also have a non-integer dimension, *e.g.* a fractal dimension of 1.4 is somewhere in between the familiar one-dimensional space of lines and curves and the two-dimensional space of planes and surfaces. Fractals are generated mathematically using an iterative process, such as the fractal fern on the title page. Fractal-like objects are readily found in nature, *e.g.* broccoli plants, fern leaves, blood vessels, mountains.

The Cantor Set is a simple fractal constructed by starting with the line segment $[0, 1]$, removing the open interval $(1/3, 2/3)$ from the middle third of the line segment, and then successively removing the open middle thirds of the subsequent line segments, see Figure 21.

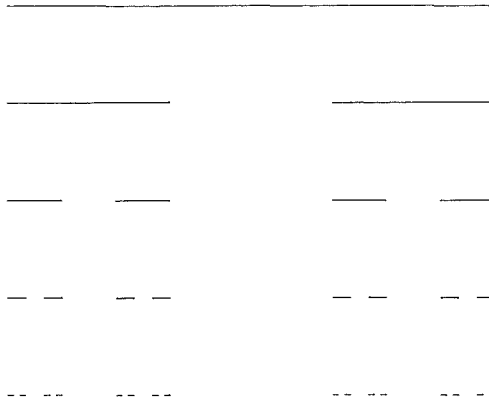


Figure 21: The first few iterations in the construction of the Cantor Set. First the open interval $(1/3, 2/3)$ is removed, then the open middle third of each remaining segment, and so on.

When this is iterated an infinite number of times, the Cantor Set C_∞ is produced. This can be difficult to visualise, but the result is an infinite number of points (which have position but no length) each separated by gaps of different widths. The Cantor Set is a simple fractal, because it has fine structure at arbitrarily small scales, and because it is self-similar. Smaller and smaller portions of the set can be magnified repeatedly and they would still show a complex pattern of points separated by gaps of different widths. The set is exactly self-similar, containing small copies of itself at all scales. If the portion of the set from 0 to $1/3$ is enlarged by a factor of 3, the Cantor Set is obtained again. This exact self-similarity is only characteristic of the simplest fractals, while more complex fractals are only approximately self-similar.

C_∞ has no length, since the sum of the length of the intervals removed at each iteration is one:

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \dots + \frac{2^n}{3^{n+1}} + \dots = \frac{1}{3}(1 + \frac{2}{3} + \frac{4}{9} + \dots) = 1$$

Therefore the dimension of the Cantor Set is not 1. But it cannot be zero-dimensional either, because it consists of an infinite number of points. The dimension of exactly self-similar fractals is described by a “similarity dimension”. If a self-similar set is made up of m copies of itself when scaled down by a factor of r , then its similarity dimension d is given by $m = r^d$ or $d = \ln m / \ln r$. Each half of the Cantor Set contains a copy of the original, scaled down by a factor of 3, so the set has 2 copies of itself when scaled down by a factor of 3, and so its dimension is $d = \ln 2 / \ln 3 \approx 0.63$. The definition of the dimension of a fractal can be generalised to include fractals that are not exactly self-similar; for further details see Strogatz [1994].

Conclusion

This has been a basic introduction to some of the concepts of maps, chaos, and fractals using one-dimensional discrete dynamical systems. There is a wealth of mathematics underlying these concepts which draws together different areas such as analysis (definitions of fixed points and periodic orbits), calculus (analysis of stability of fixed points), and geometry (structure of strange attractors and fractals). The two related subjects of chaotic dynamics and fractal geometry have made an enormous impact on the mathematical modeling of physical phenomena.

Iterated maps such as the logistic map provide fascinating examples of how complex chaotic behaviour can arise even in relatively simple deterministic systems, which can be readily studied using computers and mathematical software. All the graphs and diagrams in this report were prepared by writing Matlab code except the diagrams for the period three implies chaos proof. Chaos theory provides a fascinating glimpse into a subject that, while part of the frontier of mathematical research, also has ideas and mathematics accessible to undergraduate students.

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Appendix: Sample Matlab Files

Logistic Bifurcation Diagram

```
% Bifurcation diagram of the logistic map
% Author: Phillipa Williams
% Date: 27/12/06

clear all
close all
clc

N = 400;           % maximum number of iterations
P = 100;           % number of iterations to discard
M = N-(P-1);       % starting point in x row vector for plotting
a = 2.5;           % starting value of parameter r
b = 4.0;           % final value of parameter r
step = 0.001;      % step for loop of r values

for r = a:step:b    % outer loop to step through r values
    x = 0.4;         % initialise x_0 value
    xo = x;          % assign first x value

    for I = 1:N      % inner loop to iterate logistic map
        xn = r*xo*(1-xo); % logistic mapping
        x = [x xn];    % prepares array of x values from
iterations          % updates xo value
        xo = xn;
    end

    % plots current r value against final iterations of logistic map
    plot(r*ones(1,P),x(M:N),'.','MarkerSize',1,'Color','k')
    hold on
end

% specifies axes and labels for plot
fsize = 26;         % sets font size
axis([a b 0 1])     % sets length of axes
set(gca,'box','on') % draws a box around
plot
set(gca,'xtick',[a:0.5:b],'FontSize',fsize)
set(gca,'ytick',[0:0.2:1],'FontSize',fsize)
xlabel('r','FontSize',fsize) % x-axis label
ylabel('x','FontSize',fsize) % y-axis label
hold off

% Exports Figure 1 with 600 dpi resolution using the EPS graphics
format
print -f1 -r600 -deps logisticbifur2
```

Logistic Cobweb Diagram

```
% Cobweb iteration for the logistic map
% Author: Phillipa Williams
% Date: 28/12/06

close all;
clear;
clc;

N=150;
M=N/2;
x=zeros(1,N);          % initialise matrix
x(1)=0.2;               % initialise x_0 value
r=3.48;                 % specify r value

% computes logistic map values
for I=1:N
    x(I+1)=r*x(I)*(1-x(I));
end

hold on

% plots the first line from (x1,0) to (x(1),x(2))
line([x(1) x(1)], [0 x(2)], 'Color', 'r')

for J=1:N-1

    % plots a line from (x(n),x(n+1)) to (x(n+1),x(n+1))
    line([x(J) x(J+1)], [x(J+1) x(J+1)], 'Color', 'r')
    % plots a line from (x(n+1),x(n+1)) to (x(n+1),x(n+2))
    line([x(J+1) x(J+1)], [x(J+1) x(J+2)], 'Color', 'r')

end

% plots the final line from (x(N),x(N+1)) to (x(N+1),x(N+1))
line([x(N) x(N+1)], [x(N+1) x(N+1)], 'Color', 'r')

fplot(@(x)r*x*(1-x), [0 1], 'b');    % plots logistic map as a curve
fplot(@(x)x, [0 1], 'k');           % plots the line y=x

% specifies axes and labels for plot
axis([0 1 0 1.2])
set(gca, 'box', 'on')
set(gca, 'xtick', [0:0.2:1], 'FontSize', 26)
set(gca, 'ytick', [0:0.2:1], 'FontSize', 26)
xlabel('x', 'FontSize', 26)
ylabel('f(x)', 'FontSize', 26)
hold off

% To export Figure 1 with 600 dpi resolution using the EPS graphics
format
print -f1 -r600 -deps logisticweb2c
```

Barnsley Fern (title page illustration)

```
% Barnsley's fern fractal image
% Author: Phillipa Williams
% Date: 28/12/06

clear all
close all
clc

% initialise variables and matrices
N=100000;           % specifies number of points to plot fern
P=zeros(N,2);       % initialise matrix P
P(1,:)=[0.5,0.5];    % initialise first row of P

% performs iterations to create fern using the
% random function to generate probabilities
% and calling the affine function which contains
% the affine transformations
for k=1:N-1
    r=rand;
    if r<.05;
        P(k+1,:)=affine(P(k,:),0,0,0,0,0.2,0);
    elseif r<.86;
        P(k+1,:)=affine(P(k,:),0.85,0.05,0,-0.04,0.85,1.6);
    elseif r<.93;
        P(k+1,:)=affine(P(k,:),0.2,-0.26,0,0.23,0.22,1.6);
    else
        P(k+1,:)=affine(P(k,:),-0.15,0.28,0,0.26,0.24,0.44);
    end
end

hold on
plot(P(:,1),P(:,2),'.','MarkerSize',1,'Color','k'); % plots the fern
axis off                                           % removes axes
hold off

% To export Figure 1 with 600 dpi resolution using the
% EPS graphics format to a file named fern
print -f1 -r600 -deps fern

function[F]=affine(P,a,b,c,d,e,f)

% This function file gives the four affine transformations
% of the form affine(x,y) = (a*x+b*y+c, d*x+e*y+f)
% which are iterated in the main loop to produce the fern

F=zeros(1,2);
F(1)=a*P(1)+b*P(2)+c;
F(2)=d*P(1)+e*P(2)+f;

end
```